The Dirac Equation and General Linear Transformations of Coordinate Systems

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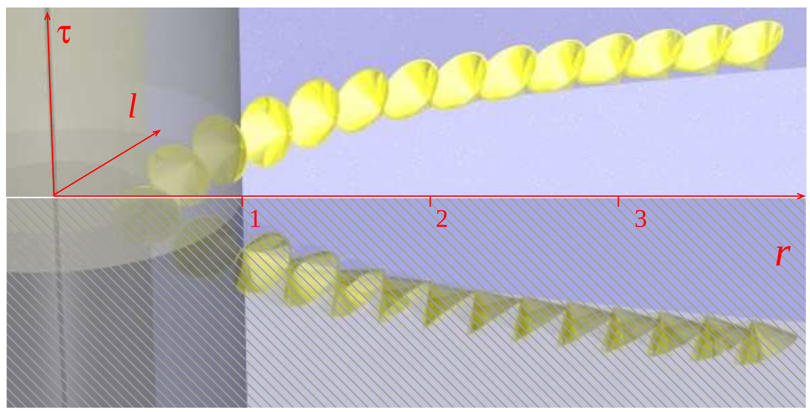

Abstract

The spinor representation of the Lorentz group does not accept simple generalization with the group $GL(4,\mathbb{R})$ of general linear coordinate transformations. The Dirac equation may be written for an arbitrary choice of a coordinate system and a metric, but the covariant linear transformations of the four-component Dirac spinor exist only for isometries. For usual diagonal Minkowski metric the isometry is the Lorentz transformation. On the other hand, it is possible to define the Dirac operator on the space of anti-symmetric (exterior) forms, and in such a case the equation is covariant for an arbitrary general linear transformation. The space of the exterior forms is sixteen-dimensional, but usual Dirac equation is defined for four-dimensional complex space of Dirac spinors. Using suggested analogy, in present paper is discussed possibility to consider the space of Dirac spinors as some "subsystem" of a bigger space, where the group $GL(4,\mathbb{R})$ of General Relativity acts in a covariant way.

For such purposes in this article is considered both Grassmann algebra of complex anti-symmetric forms and Clifford algebra of Dirac matrices. Both algebras have same dimension as linear spaces, but different structure of multiplication. The underlying sixteen-dimensional linear space also may be considered either as space of complex 4×4 matrices, or as space of states of two particles: the initial Dirac spinor and some auxiliary system. It is shown also, that such approach is in good agreement with well known idea to consider Dirac spinor as some ideal of Clifford algebra. Some other possible implications of given model are also discussed.

1 Introduction

1.1 Klein-Gordon and Dirac Equations

Let us consider an operator of second order

$$\Box = -\sum\nolimits_{kj} g_{kj} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j}. \tag{1.1}$$

For Minkowski metric g of (flat) spacetime the operator Eq. (1.1) is the d'Alembert (wave) operator and components ψ of a relativistic field for a particle with mass m satisfy to a Klein–Gordon equation (in the system of units $\hbar = 1$, $c = 1$).

$$(\Box - m^2)\psi = 0. \tag{1.2}$$

For a relativistic particle with spin 1/2, the Klein-Gordon operator should be rewritten as [1]

$$\Box = \mathcal{D}^2, \quad \Box - m^2 = (\mathcal{D} - m)(\mathcal{D} + m), \tag{1.3}$$

where

$$\mathcal{D} = \sum_{k=0}^{3} i\gamma^k \frac{\partial}{\partial x_k} \tag{1.4}$$

is Dirac operator and γ_k are four 4×4 complex matrices with property

$$\gamma^j\gamma^k + \gamma^k\gamma^j = 2g_{kj}\mathbf{1}, \tag{1.5}$$

where $\mathbf{1}$ is unit matrix.

The Dirac equation for a four-component spinor $\psi \in \mathbb{C}^4$ may be written as

$$\mathcal{D}\psi = m\psi, \quad \psi \in \mathbb{C}^4, \quad \mathcal{D} = \sum_{k=0}^{3} i\gamma^k \frac{\partial}{\partial x_k} \tag{1.6}$$

and due to Eq. (1.3) any component of ψ also satisfies to the Klein-Gordon equation Eq. (1.2).

Really the factorization $\mathcal{D}^2 = \Box$ used in Eq. (1.3) together with Eq. (1.5) may be written for any metric g (and dimension) [3].

1.2 Two-component Spinors

It is also convenient to consider construction of the four-component Dirac spinors as a pair of two-component (Pauli[1]) spinors [2]. Here we are again considering

[1]The two-component spinors also are called "Weyl spinors," "half-spinors" or simply "spinors" — in the last case the four-component spinors are called "bispinors."

factorization of the Klein–Gordon equation, but instead of the formal decomposition Eq. (1.3), it is possible to write two equations for Pauli spinors $\eta, \xi \in \mathbb{C}^2$

$$i(\frac{\partial}{\partial x_0} + \sum_{k=1}^{3} \sigma_k \frac{\partial}{\partial x_k})\eta = m\xi, \quad i(\frac{\partial}{\partial x_0} - \sum_{k=1}^{3} \sigma_k \frac{\partial}{\partial x_k})\xi = m\eta \tag{1.7}$$

with 2×2 Pauli matrices

$$\sigma_1 \equiv \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \equiv \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \equiv \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.8}$$

From such a point of view, the four-component Dirac spinor may be considered as a composition of two Pauli spinors

$$\psi = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \tag{1.9}$$

and then two equations Eq. (1.7) are equivalent to one Dirac equation Eq. (1.6) with 4×4 matrices [2]

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad (k = 1, 2, 3) \tag{1.10}$$

(where 0 and 1 are also 2×2 matrices). Of course, the matrices Eq. (1.10) are satisfying Eq. (1.5) for Minkowski metric.

1.3 Dirac–Kähler Equation

Factorization of d'Alembertian Eq. (1.3) using Dirac operator, has some resemblance with decomposition of Laplace–Beltrami operator in the Hodge theory [4]

$$\Delta = dd^\star + d^\star d = -(d - d^\star)^2, \quad d^\star = \pm \star d \star, \quad d^2 = {d^\star}^2 = 0 \tag{1.11}$$

and it is possible to introduce representation of the Dirac operator on space of anti-symmetric (exterior) forms [3, 8]. In such a case an analogue of the Dirac equation should be written as

$$\check{\mathcal{D}}\Upsilon = m\Upsilon, \quad \check{\mathcal{D}} \equiv (d - d^\star) \tag{1.12}$$

where $\Upsilon \in \Lambda$ is some anti-symmetric form. Such a kind of Dirac equation is widely used since introduction by Landau and Ivanenko at 1928 and Kähler at 1962 [5, 6, 7]. It should be mentioned, that because $d : \Lambda^p \to \Lambda^{p+1}$ and $d^\star : \Lambda^p \to \Lambda^{p-1}$, the Dirac–Kähler operator $d - d^\star$ does not respect the order p of a form, unlike the Laplace–Beltrami operator $\Delta : \Lambda^p \to \Lambda^p$.

1.4 Covariance

A multi-component quantum field should be *covariant* with respect to coordinate transformations [1], *e.g.*, components of the quantum field $u(x)$ after change of coordinate system $L : x \to x'$ also are changed $u' = \mathfrak{S}_L u$, where $\mathfrak{S}_L$ is some representation of group of coordinate transformations acting on the space of multi-component wave vectors.

Different groups of acceptable coordinate transformations may cause particular limitations to a model. Say, the two-component spinors discussed above are covariant not only with respect to group $SO(3)$ of transformations of non-relativistic space (3D rotation), but also with respect to Lorentz group $SO(3,1)$. On the other hand, the Pauli spinors are not covariant with respect to a bigger group $O(3,1)$ with the time reflection and it may be considered as a reason of the necessity to consider the four-component Dirac spinors [2].

The Dirac spinors are not covariant with respect to group $GL(4,\mathbb{R})$ of all possible coordinate transformations. On the other hand, the anti-symmetric forms in Eq. (1.12) and operators $d, d^\star$ used for construction Eq. (1.11) of the Dirac–Kähler operator are covariant with respect to the general linear group $GL(4,\mathbb{R})$ of coordinate transformations, it is well known property of the differentials and tensor fields.

The transition from space of two-component spinors to the bigger space of four-component spinors let us consider covariance with respect to $O(3,1)$ instead of $SO(3,1)$ and $O(3)$. Formally, the transition from space of four-component spinors to the bigger sixteen-dimensional space of anti-symmetric forms made the Dirac equation covariant with respect to the group $GL(4,\mathbb{R})$ of all coordinate transformations.

It is possible also to consider a space of exterior forms with complex coefficients and use other representations with sixteen-dimensional linear spaces, for example the space of 4×4 matrices, or tensor product of two four-dimensional spaces. The last example also formally corresponds to consideration of a compound quantum system $\mathbb{C}^4 \otimes \mathbb{C}^4$ with two particles described by the Dirac spinors. In such a case relation with the initial Dirac equation is not so simple, as in the example with transition between two- and four-component spinors.

1.5 Outline

The present article is devoted to consideration of such $GL(4,\mathbb{R})$-covariant extension of Dirac equation with a sixteen-dimensional linear space, together with description of relation with usual Dirac equation for four-components spinors and some implications of such generalization.

It is extension and development of an earlier work [9]. In Sec. 2 and Sec. 3 are described necessary methods from theory of Clifford algebras and Spin groups. Exterior forms and Hodge theory are revisited in Sec. 4. Transformation properties of Dirac equation are discussed in Sec. 5. In Sec. 6 are discussed some applications of

group $GL(4,\mathbb{R})$ of general linear coordinate transformations to General Relativity, because it is useful for justification of suggested approach with generalization of Dirac equation.

2 Clifford Algebra of a Quadratic Form

2.1 Definition of Clifford Algebra

The decomposition of the wave operator Eq. (1.1) in Sec. 1 follows to a standard procedure of representation of a quadratic form as the square of an element of some algebra [3, 10]. If there is n-dimensional vector space V with a quadratic form $g(v) \equiv g(v,v)$, $v \in V$, some algebra $\mathfrak{Cl}$ and n elements (generators) $\mathsf{e}_k \in \mathfrak{Cl}$, it is possible to consider the linear embedding $\iota : V \to \mathfrak{Cl}$, $\iota(v) = \sum v_k \mathsf{e}_k \in \mathfrak{Cl}$, where v_k are components of the vector v in some basis. Let us denote $\iota(v)$ simply as $\mathtt{v}$. In such a case it is reasonable to look for expression of the quadratic form g as the square of $\mathtt{v} \in \mathfrak{Cl}$

$$-g(v)\mathtt{1} = \mathtt{v}^2, \quad \mathtt{v} \equiv \iota(v), \tag{2.1}$$

where $\mathtt{1}$ is the unit of the algebra, and the minus sign is due to some traditions [3] ("hypercomplex numbers"), but may be omitted in some accounts [10]. It is simply to check, that from Eq. (2.1) directly follows an equivalent of Eq. (1.5)

$$\{\mathsf{e}_k, \mathsf{e}_j\} \equiv \mathsf{e}_k \mathsf{e}_j + \mathsf{e}_j \mathsf{e}_k = -2 g_{kj} \mathtt{1}. \tag{2.2}$$

Due to Eq. (2.2) n generators e_k may produce up to 2^n linearly independent products and it is the maximum possible dimension of the Clifford algebra generated by these elements [3].

Subspaces generated by products of k different generators are denoted here $\mathfrak{Cl}^{(k)}$

$$\mathfrak{Cl}^{(k)} = \operatorname{span}\{\mathsf{e}_{i_1} \cdots \mathsf{e}_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\}, \quad \dim \mathfrak{Cl}^{(k)} = \binom{n}{k}, \tag{2.3}$$

and are called sometime *k-multivectors*. It is possible to decompose the Clifford algebra in the direct sum of $n+1$ such subspaces $\mathfrak{Cl} = \mathfrak{Cl}^{(0)} \oplus \mathfrak{Cl}^{(1)} \oplus \cdots \oplus \mathfrak{Cl}^{(n)}$.

2.2 Grassmann Algebra

A particular case [3] is the trivial, degenerate quadratic form $g(v) \equiv 0$, $\forall v \in V$. For such a case Eq. (2.2) correspond to the definition of *the Grassmann algebra*. In such a case product of elements $\mathtt{ab}$ often denoted $\mathtt{a} \wedge \mathtt{b}$ and instead of Eq. (2.1) and Eq. (2.2) we have

$$\mathtt{v}^2 = \mathtt{v} \wedge \mathtt{v} = 0; \qquad \mathsf{e}_k \wedge \mathsf{e}_j + \mathsf{e}_j \wedge \mathsf{e}_k = 0 \quad (\forall j,k). \tag{2.4}$$

The Grassmann algebras devote separate consideration, see Sec. 4. For any quadratic form g it is possible to use decomposition $V = R \oplus R^{\perp}$, with degenerate

and nondegenerate subspaces R and $R^\perp$ respectively [3], and so it is reasonable to describe classification of the Clifford algebras with nondegenerate quadratic forms.

Sometime it is also convenient for complex Clifford algebras with even dimension and the diagonal form (like Eq. (2.10) below) to consider new set of generators

$$\mathtt{a}_j = \frac{1}{2}(\mathtt{e}_j + i\mathtt{e}_{m+j}), \quad \mathtt{a}_j^* = \frac{1}{2}(\mathtt{e}_j - i\mathtt{e}_{m+j}), \quad j = 1, \ldots, m. \tag{2.5}$$

Then both subsets with m elements $\mathtt{a}_j$ or $\mathtt{a}_j^*$ generate Grassmann subalgebras of the initial Clifford algebra. Together with usual anticommutation Eq. (2.4) of all elements in each algebra, there is specific relations between pair of elements from different subsets, *i.e.*

$$\{\mathtt{a}_j, \mathtt{a}_k\} = \{\mathtt{a}_j^*, \mathtt{a}_k^*\} = 0, \quad \{\mathtt{a}_j, \mathtt{a}_k^*\} = 2\delta_{jk}. \tag{2.6}$$

The Eq. (2.6) coincide with canonical anticommutation relations (CAR) for annihilation and creation operators for fermions. Choice of representation of $\mathtt{e}_j$ with tensor product of Pauli matrices (see Eq. (2.11) below) corresponds to the usual Jordan–Wigner formalism [11]

$$\begin{aligned} \mathtt{a}_k &= \underbrace{1 \otimes \cdots \otimes 1}_{m-k-1} \otimes a \otimes \underbrace{\sigma_z \otimes \cdots \otimes \sigma_z}_{k}, \\ \mathtt{a}_k^* &= \underbrace{1 \otimes \cdots \otimes 1}_{m-k-1} \otimes a^+ \otimes \underbrace{\sigma_z \otimes \cdots \otimes \sigma_z}_{k}, \end{aligned} \tag{2.7}$$

where 1 is 2×2 unit matrix and

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.8}$$

So we have two different representations of the real Grassmann algebra with m generators in the same algebra of $2^m \times 2^m$ real matrices.

2.3 Nondegenerate Quadratic Forms

For a nondegenerate quadratic form it is always possible to choose *the normalized basis*. In such a basis the matrix of the quadratic form is diagonal $g_{jk} = \pm\delta_{jk}$ and it is convenient to use such forms for classifications of *the real* Clifford algebras [3, 10, 12]. Here the algebra for the quadratic form with l positive and m negative terms is denoted as $\mathfrak{Cl}(l,m)$. For a complex vector space the signs do not matters, because they always may be changed using multiplication on i and so for Clifford algebra with complex coefficients here is used notation $\mathfrak{Cl}(n,\mathbb{C})$.

For the complex case classification of the Clifford algebras $\mathfrak{Cl}(n,\mathbb{C})$ is simpler. For even $n = 2m$ there is an isomorphism with the algebra of $2^m \times 2^m$ complex matrices and for odd case $2m+1$ with the direct sum of two matrix algebras [3]

$$\mathfrak{Cl}(2m,\mathbb{C}) \cong M(2^m,\mathbb{C}); \quad \mathfrak{Cl}(2m+1,\mathbb{C}) \cong M(2^m,\mathbb{C}) \oplus M(2^m,\mathbb{C}). \tag{2.9}$$

The generators e_k of the Clifford algebra for even dimension $\mathfrak{Cl}(2m,\mathbb{C})$ satisfying Eq. (2.2) for Euclidean form

$$\{\mathsf{e}_k, \mathsf{e}_j\} \equiv \mathsf{e}_k\mathsf{e}_j + \mathsf{e}_j\mathsf{e}_k = 2\delta_{kj}\mathbb{1}. \tag{2.10}$$

may be expressed using Pauli matrices as [3, 11]

$$\begin{aligned} \mathsf{e}_{2k} &= \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{m-k-1}\otimes\sigma_x\otimes\underbrace{\sigma_z\otimes\cdots\otimes\sigma_z}_{k}, \\ \mathsf{e}_{2k+1} &= \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{m-k-1}\otimes\sigma_y\otimes\underbrace{\sigma_z\otimes\cdots\otimes\sigma_z}_{k}, \end{aligned} \tag{2.11}$$

where $k = 0,\ldots,m-1$ and $\mathbb{1}$ is 2×2 unit matrix.

The algebra $\mathfrak{Cl}(2m+1,\mathbb{C})$ may be considered as a subspace of $\mathfrak{Cl}(2m+2,\mathbb{C})$ with a reduced set of generators Eq. (2.11) (without e_{2m+2}). It is also convenient to change σ_x to σ_z in the last generator

$$\mathsf{e}_{2m+1} = \underbrace{\sigma_z\otimes\cdots\otimes\sigma_z}_{2m+1}, \tag{2.12}$$

because in such a case the structure $\mathfrak{Cl}(2m+1,\mathbb{C}) \cong \mathfrak{Cl}(2m,\mathbb{C}) \oplus \mathfrak{Cl}(2m,\mathbb{C})$ is more transparent.

For real Clifford algebras $\mathfrak{Cl}(l,m)$ the classification is more difficult. Different cases may be arranged using a 8×8 table ($l,m = 0,\ldots,7$) and recurrent structure of $\mathfrak{Cl}(8k+l, 8j+m)$ [12] instead of only two different structures $\mathfrak{Cl}(2k+l)$, $l = 0,1$ in complex case. Anyway, all the real Clifford algebras may be represented either as a some matrix algebra with real, complex, quaternion coefficients, or direct sum of two such algebras.

The space of spinors — is a space of representations of the Clifford algebras (and Spin groups, see Sec. 3 below). So, if a Clifford algebra is isomorphic with algebra of $N\times N$ (real, complex, quaternion) matrices, the spinors are N-dimensional (real, complex, quaternion[2]) linear spaces.

It should be mentioned, that $\mathfrak{Cl}(l,m)$ for a particular choice of l and m may contain an element $\mathtt{i} \in \mathfrak{Cl}$ commuting with all other elements of algebra and with property $\mathtt{i}^2 = -\mathbb{1}$. Formally it should be distinguished from a complex Clifford algebra, because for construction was used the real linear space and so $\mathtt{i}$ here is not a number ("scalar"), but an element of the real algebra. For example

$$\mathfrak{Cl}(3,0) \cong \mathfrak{Cl}(1,2) \cong M(2,\mathbb{C}), \quad \mathfrak{Cl}(4,1) \cong \mathfrak{Cl}(2,3) \cong M(4,\mathbb{C}), \tag{2.13}$$

are real Clifford algebras (see [12]) and

$$\mathfrak{Cl}(2,\mathbb{C}) \cong M(2,\mathbb{C}), \quad \mathfrak{Cl}(4,\mathbb{C}) \cong M(4,\mathbb{C}) \tag{2.14}$$

are complex Clifford algebras.

[2] Quaternions are not commutative, so formally instead of linear space should be used left quaternion module.

2.4 Transformation Properties

Let us suggest, that we have built a Clifford algebra for some quadratic form g and want to rewrite the relations Eq. (2.2) for a new set of elements

$$\mathsf{e}'_k = \sum_j A_{kj}\mathsf{e}_j. \tag{2.15}$$

Then the set of equations Eq. (2.2) for the new elements is

$$-2g'_{kl} = \{\mathsf{e}'_k, \mathsf{e}'_l\} = \sum_{jm}\{A_{kj}\mathsf{e}_j, A_{lm}\mathsf{e}_m\} = -2\sum_{jm} A_{kj}A_{lm}g_{jm}, \tag{2.16}$$

and using a matrix notation last equation may be rewritten as

$$g' = AgA^T, \tag{2.17}$$

where A is the matrix of coefficients A_{kj} of transformation Eq. (2.15) and A^T is the transposed matrix. The Eq. (2.17) coincides with the formula for change of the metric after transition (described by the matrix A) to other basis in the space V.

So "algebraic square root" Eq. (2.1) of an arbitrary (not necessary diagonal) quadratic form g' may be found using diagonalization of given form and equation Eq. (2.17). It is necessary first to convert the quadratic form to a sum of squares using appropriate transformation of a basis in V and to find a Clifford algebra for the diagonal form. Next, with n generators e_k of given algebra, it is possible using Eq. (2.15) to build the a set with n elements e'_j satisfying Eq. (2.2) for the initial quadratic form g'.

Such a technical way to describe a construction of the spinors formally includes few objects: the vector space V with the metric g, a diagonal metric g_0 and a matrix A ("vielbein") of transition from normalized form $g(v, w) = g_0(Av, Aw)$. It is relevant to *a tetrad (vierbein) formalism of the General Relativity* (see Sec. 5.1 and references therein).

Really the tetrad formalism does not necessary suggest the choice of g_0 in a diagonal form, it may be any other form convenient for description of the Clifford algebra. For example, it is possible to choose "an isotropic basis" Eq. (2.5). The main property of a tetrad representation — is using some fixed form g_0 and necessity of "vielbein" A for transition to concrete metric g. An example with isotropic tetrads is Newman–Penrose formalism [13, 20].

On the other hand, for complete description of a Clifford algebra $\mathfrak{Cl}$ with a quadratic form g, it is enough to give only a vector space V with the metric g, the algebra $\mathfrak{Cl}$ and the embedding $\iota : V \to \mathfrak{Cl}$, $g(x) = -\iota^2(x)$ used in the definition Eq. (2.1). In such a case, A and g_0 look as redundant objects. Covariant character of such definition may be more clear from construction with exterior (Grassmann) algebra discussed below in Sec. 4.

2.5 Automorphisms

It should be mentioned, that Eq. (2.15) might not be considered as some transformation of the *whole* Clifford algebra. The matrix A describes transformation of the vector space V and an n-dimensional subspace of the algebra defined as embedding $\iota(V)$ or as the linear span of the elements $\mathtt{e}_k$, but it is not possible without additional suggestions to apply it to an arbitrary element of the algebra like $\mathtt{e}_j\mathtt{e}_k$, $\mathtt{e}_j\mathtt{e}_k\mathtt{e}_l$, *etc.*

One method to extend such transformations from the generators to the whole algebra, is to use *automorphisms*, *i.e.* the linear maps to itself $\alpha : \mathfrak{Cl} \to \mathfrak{Cl}$ with property

$$\alpha(\mathtt{a}\,\mathtt{b}) = \alpha(\mathtt{a})\,\alpha(\mathtt{b}) \tag{2.18}$$

for any elements $\mathtt{a}, \mathtt{b}$ of the algebra.

A particular example is *the internal automorphism*, defined as

$$\alpha_{\mathtt{h}}(\mathtt{a}) = \mathtt{h}\,\mathtt{a}\,\mathtt{h}^{-1}, \tag{2.19}$$

where $\mathtt{h}$ is an arbitrary invertible element of the algebra.

Due to Eq. (2.2) any automorphism of a Clifford algebra saves the same quadratic form for the new set of generators $\tilde{\mathtt{e}}_k = \alpha(\mathtt{e}_k)$

$$\{\tilde{\mathtt{e}}_k, \tilde{\mathtt{e}}_j\} = \alpha(\mathtt{e}_k)\alpha(\mathtt{e}_j) + \alpha(\mathtt{e}_j)\alpha(\mathtt{e}_k) = \alpha(\mathtt{e}_k\mathtt{e}_j + \mathtt{e}_j\mathtt{e}_k) = -2g_{kj}\alpha(1) = -2g_{kj}\mathtt{1}. \tag{2.20}$$

So, it is an important question, if the map Eq. (2.15) corresponds to some automorphism of the algebra. It is useful to consider two different examples.

1. It was shown that the Grassmann algebra is equivalent to a Clifford algebra with a degenerated quadratic form $g \equiv 0$ Eq. (2.4). Any coordinate transformations saves such a trivial form and for any A Eq. (2.15) may be extended to an automorphism of the whole algebra, it corresponds to usual covariant transformations of anti-symmetric tensors $\Lambda = \bigoplus \Lambda^k$ with ranges $0 \le k \le n$ (*e.g.* see Eq. (4.2) in Sec. 4). The automorphism may not be internal, because all elements of Grassmann algebra, except $\mathtt{1}$, do not have inverse.

2. For the Clifford algebra a with nondegenerate form g only *isometries* A saves the quadratic form (by definition). So only for isometries Eq. (2.15) could be expressed as some automorphism, and really always exists *an internal isomorphism* with necessary property, *i.e.* for any isometry A exists element $\mathtt{h}_A \in \mathfrak{Cl}$:

$$\iota(Av) = \mathtt{h}_A\,\iota(v)\,\mathtt{h}_A^{-1} \quad (v \in V), \tag{2.21}$$

i.e., instead of Eq. (2.15) for isometry A it is possible to write

$$\mathtt{e}'_j = \mathtt{h}_A\,\mathtt{e}_j\,\mathtt{h}_A^{-1} \quad (\forall j,\ \text{no summation}), \tag{2.22}$$

but such element $\mathtt{h}_A$ exists if and only if $g(Av) = g(v)$. It corresponds to construction of *Spin group* for given Clifford algebra [3, 10].

3 Spin Groups

3.1 Reflections and Rotations

A simple example of the isometry is *the reflection*, R_v, defined by some $v \in V$, $g(v) \neq 0$ as

$$R_v : x \mapsto x - 2\frac{g(x,v)}{g(v)}v. \tag{3.1}$$

Any isometry of n-dimensional space may be written as composition of few reflections [3, 15].

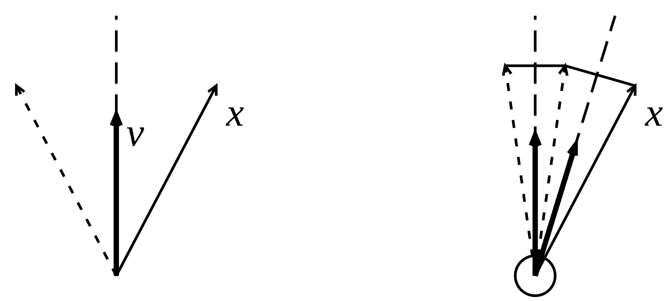

Figure 1: Example of reflection and presentation of rotation as two reflections for two-dimensional plane

It may be checked directly [3], that the reflection may be simply written by using representation of the vector as an element of the Clifford algebra

$$R_v : \mathtt{x} \mapsto \mathtt{v}\,\mathtt{x}\,\mathtt{v}^{-1}. \tag{3.2}$$

It is also possible to consider only elements with $g(v) = \pm 1$, because for any number $\lambda \neq 0$, the vectors v and λv represent the same reflection.

3.2 Spin, Pin and Spoin Groups

In both Euclidean and pseudo-Euclidean case, any element of the group $SO(l,m)$ may be represented as a composition of *even* number of reflections, and so, if to consider *the group* $Spin(l,m)$*, consisting of all possible even products* $\mathtt{h} = \mathtt{v}_1 \cdots \mathtt{v}_{2k} \in \mathfrak{Cl}(l,m)$, for $v_j \in V$, $g(v_j) = \pm 1$, we have representation of the group $SO(l,m)$ as

$$\mathtt{x} \mapsto \mathtt{h}\,\mathtt{x}\,\mathtt{h}^{-1}, \quad \mathtt{h} \in Spin(l,m) \tag{3.3}$$

and because $\mathtt{h}$ and $-\mathtt{h}$ correspond to the same transformation, it is $2 \to 1$ representation of $SO(l,m)$ group. Similarly, *all products with an arbitrary (not necessary even) number of elements* $\mathtt{v}_j$ *produce group* $Pin(l,m)$, it is 2→1 representation of $O(l,m)$ group [3, 10].

Let us consider *the linear span of all possible products with even number of generators*. The subspace is closed under multiplication, includes unit and so may be considered as *a new Clifford algebra*, generated by $n-1$ generators $\acute{\mathtt{e}}_j = \mathtt{e}_j\mathtt{e}_n$, $j = 1,\ldots,n-1$. Spin group also may be considered using this Clifford algebra with smaller dimension. Such "economical" representation of Spin group sometime called *the Spoin group*, but in such a case, instead of internal automorphism Eq. (3.3) should be used changed expression

$$\mathtt{x} \mapsto \mathtt{h}\,\mathtt{x}\,\mathtt{h}'^{-1}, \quad \mathtt{h} \in Spoin(l,m) \tag{3.4}$$

where $\mathtt{h}'$ is specific "Clifford conjugation" [3].

The relation of Spoin and Spin groups is important for discussion about representations of the Lorentz group with Pauli and Dirac matrices in Sec. 5.1.

4 Grassmann Algebra of Exterior Forms

4.1 Forms and Tensors

Let us recall some standard definitions and properties [4, 8, 16, 20]. The *tensor of type* (r,s) may be written using a basis e_j in an n-dimensional vector space V and e^j in the dual space V^* (linear functions on V) as

$$T = \sum T^{i_1\ldots i_r}_{j_1\ldots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}. \tag{4.1}$$

For a change of the basis described by the matrix A, $e_i = \sum_j A^j_i e'_j$ such a tensor is transformed as

$$T'^{i_1\ldots i_r}_{j_1\ldots j_s} = \sum A^{i_1}_{k_1} \cdots A^{i_r}_{k_r} B^{m_1}_{j_1} \cdots B^{m_s}_{j_s} T^{k_1\ldots k_r}_{m_1\ldots m_s} \tag{4.2}$$

where $B = A^{-1}$ is the inverse of the matrix A [16].

An exterior s-form $\omega \in \Lambda^s(V)$ is *the totally antisymmetric tensor* of type $(0,s)$. Any tensor of type $(0,s)$ may be considered as multilinear functional and s-form is a specific one, antisymmetric in every pair of arguments

$$\omega(v_1,\ldots,v_i,\ldots,v_j,\ldots,v_s) = -\omega(v_1,\ldots,v_j,\ldots,v_i,\ldots,v_s), \quad \forall i,j. \tag{4.3}$$

Let us consider an operation of *alternation* of a tensor of type $(0,s)$

$$\mathcal{A}T(v_1,\cdots,v_s) = \frac{1}{s!}\sum_{\pi}(-)^{\pi} T(v_{\pi(1)},\cdots,v_{\pi(s)}), \tag{4.4}$$

where π denotes all $s!$ permutations of s indexes and $(-)^\pi$ is $+1$ for an even permutation and -1 for an odd one, then $\mathcal{A}T$ is always antisymmetric tensor. The basis of space of the antisymmetric tensors may be written using an exterior (wedge) product

$$e^{j_1} \wedge \cdots \wedge e^{j_s} = \mathcal{A}(e^{j_1} \otimes \cdots \otimes e^{j_s}). \tag{4.5}$$

For arbitrary forms $\omega \in \Lambda^r$, $\vartheta \in \Lambda^s$ it may be defined

$$\omega \wedge \vartheta \equiv \mathcal{A}(\omega \otimes \vartheta), \quad \omega \wedge \vartheta = (-1)^{rs} \vartheta \wedge \omega. \tag{4.6}$$

A maximal possible range of the exterior form is $s = n$ and the space of all exterior forms is denoted as

$$\Lambda(V) = \bigoplus_{j=0}^{n} \Lambda^j(V), \quad \dim \Lambda(V) = 2^n, \quad \dim \Lambda^j(V) = \binom{n}{j}. \tag{4.7}$$

It is possible to consider the tensor fields on an n-dimensional manifold. For an exterior form ω it is also possible to introduce *exterior differentiation* $d\omega$, a linear operator from space of s-forms to $(s+1)$-forms $d : \Lambda^s(V) \to \Lambda^{s+1}(V)$ with properties

1. For function, *i.e.* $f \in \Lambda^0(V)$, it is usual *total differential.*
2. For $\omega \in \Lambda^r(V)$ and $\vartheta \in \Lambda^s(V)$: $d(\omega \wedge \vartheta) = (d\omega) \wedge \vartheta + (-1)^r \omega \wedge (d\vartheta)$.
3. "Poincaré d-lemma:" $d^2 = 0$.

4.2 Hodge Theory

Let us consider spaces with scalar product (metric). If some metric $<\cdot\,,\cdot>$ is introduced in $\Lambda^1(V) \cong V^*$, it is possible to extend it on whole $\Lambda(V)$. For two forms with different ranges it is zero, otherwise the norm [4]

$$<w_1 \wedge \cdots \wedge w_p \,, u_1 \wedge \cdots \wedge u_1> = \det<w_i \,, u_j>. \tag{4.8}$$

Let us consider *the Hodge operator* $\star : \Lambda^p(V) \to \Lambda^{n-p}(V)$

$$\omega \wedge \star\vartheta = <\omega \,, \vartheta> \Omega, \quad \omega, \vartheta \in \Lambda(V) \tag{4.9}$$

where $\Omega \equiv e_1 \wedge \cdots \wedge e_n$ (*volume form*). For p-form $\star\star = (-1)^{p(n-p)}$. It is possible to define $d^\star : \Lambda^p(V) \to \Lambda^{p-1}(V)$

$$d^\star = (-1)^{n(p+1)+1} \star d\star, \tag{4.10}$$

and *the Laplace–Beltrami operator*

$$\Delta = d^\star d + d\, d^\star. \tag{4.11}$$

4.3 Algebraic Approach

Representation of the Clifford and exterior calculus may be also described in more algebraic way [3, 17]. Let us return to consideration of an exterior (Grassmann) algebra, as a linear span of all possible products with n generators θ_i, if the wedge product is defined by formal equations like Eq. (2.4)

$$\theta_k \wedge \theta_j + \theta_j \wedge \theta_k = 0, \quad \omega \wedge \vartheta = (-1)^{rs} \vartheta \wedge \omega \quad (\omega \in \Lambda^r,\, \vartheta \in \Lambda^s), \tag{4.12}$$

where Λ^r is the wedge product or r generators.

Let us consider n formal operators $\delta_i : \Lambda^k \to \Lambda^{k+1}$

$$\delta_i : \theta_{j_1} \wedge \cdots \wedge \theta_{j_k} \mapsto \theta_i \wedge \theta_{j_1} \wedge \cdots \wedge \theta_{j_k}. \tag{4.13}$$

If we have some matrix g_{ij}, it is possible to define the dual operators $\delta_i^\star : \Lambda^k \to \Lambda^{k-1}$

$$\delta_i^\star : \theta_{j_1} \wedge \cdots \wedge \theta_{j_k} \mapsto \sum_{p=1}^{k} (-1)^{p-1} g_{i j_p}\, \theta_{j_1} \wedge \cdots \wedge \hat{\theta}_{j_p} \wedge \cdots \wedge \theta_{j_k}, \tag{4.14}$$

where $\hat{\theta}_{j_p}$ means, that the term θ_{j_p} must be omitted.

Both δ_i and $\delta_i^\star$ belong to a space $\mathcal{L}(\Lambda)$ of linear operators on Λ. Let us consider n operators

$$\check{\gamma}_i = \delta_i - \delta_i^\star, \quad \check{\gamma}_i \in \mathcal{L}(\Lambda), \tag{4.15}$$

then

$$\check{\gamma}_i \check{\gamma}_j + \check{\gamma}_j \check{\gamma}_i = -2 g_{ij} 1 \tag{4.16}$$

and an algebra generated by the operators $\check{\gamma}_i$ and their products — is the representation of the Clifford algebra $\mathfrak{Cl}(g)$ with the quadratic form g_{ij} in the space $\mathcal{L}(\Lambda)$ of operators on the Grassmann algebra. The underlying vector space of the Clifford and the Grassmann algebras may be identified

$$\theta_{j_1} \wedge \cdots \wedge \theta_{j_k} \longleftrightarrow \check{\gamma}_{j_1} \cdots \check{\gamma}_{j_k}, \quad j_1 < \cdots < j_k \le n. \tag{4.17}$$

It is possible also to write the formal Dirac operator [3] as some differential operator with coefficients in an algebra

$$d = \sum_{i=1}^{n} \delta_i \frac{\partial}{\partial x_i}, \quad d^\star = \sum_{i=1}^{n} \delta_i^\star \frac{\partial}{\partial x_i}, \quad \mathcal{D} = d - d^\star = \sum_{i=1}^{n} \check{\gamma}_i \frac{\partial}{\partial x_i}. \tag{4.18}$$

5 Transformation Properties of Dirac Equation

5.1 Spinor Representation of Lorentz Group

For the Lorentz signature $(3,1)$ two Clifford algebras $\mathfrak{Cl}(3,1)$ and $\mathfrak{Cl}(1,3)$ are not isomorphic, but both are subalgebras of the Dirac algebra $\mathfrak{Cl}(4,\mathbb{C})$. The algebra

$\mathfrak{Cl}(3,1)$ isomorphic with algebra of all 4×4 real matrices, it may be convenient for description of Majorana spinors, and corresponds to possibility rewrite Dirac equation using only matrices with real coefficients [2].

The spinor representation $Spin(3,1)$ of the Lorentz group is isomorphic with a classical Lie group $SL(2,\mathbb{C})$ of 2×2 complex matrices with unit determinant. Detailed and understanding discussion about this representation may be found elsewhere [13], but here it should be mentioned, that such a reduction from 4D to 2D complex space is an example of "economic" representation with Spoin group discussed in Sec. 3.2.

An even subalgbera of $\mathfrak{Cl}(3,1)$ is $\mathfrak{Cl}(3,0) \cong M(2,\mathbb{C})$, the algebra of all 2×2 complex matrices and it let us write usual representation of $Spin(3,1)$ [really, $Spoin(3,1)$] using such a matrices. On the other hand, it was noted in Sec. 3.2, that not internal isomorphism Eq. (3.3), but other transformation Eq. (3.4) should be used in such a case. For two-spinors it is usual expression [13]

$$\begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \mapsto S \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} S^*$$

or

$$t + x\sigma_x + y\sigma_y + z\sigma_z \mapsto S\,(t + x\sigma_x + y\sigma_y + z\sigma_z)\,S^* \tag{5.1}$$

with $S \in SL(2,\mathbb{C})$ and the conjugated matrix S^*. Here a vector with four real coefficients $v = (t,x,y,z)$ maps to some $v' = (t',x',y',z') = Lv$, with L is the Lorentz transformation.

On the other hand, for Dirac matrices and 4D complex space of Dirac spinors, it is possible to use expression like Eq. (3.3) with an automorphism of the algebra of 4×4 matrices

$$t\gamma^0 + x\gamma^1 + y\gamma^2 + z\gamma^3 \mapsto \mathfrak{S}\,(t\gamma^0 + x\gamma^1 + y\gamma^2 + z\gamma^3)\,\mathfrak{S}^{-1}, \quad \psi \mapsto \mathfrak{S}\psi, \tag{5.2}$$

where γ^k are four Dirac gamma matrices Eq. (1.10).

Here the matrix $\mathfrak{S}$ is a 4×4 complex matrix, *e.g.*

$$\mathfrak{S} = \begin{pmatrix} S & 0 \\ 0 & S^{*-1} \end{pmatrix}, \quad S \in SL(2,\mathbb{C}) \tag{5.3}$$

It was already mentioned, that the four-component Dirac spinor are useful for description of group $O(3,1)$ with improper transformations like a time reflection ($Pin(3,1)$ group), but transformation Eq. (5.2) also convenient for covariant description of the Dirac spinors. Let us consider the spinor space as some abstract complex space $\mathbb{C}^4$, together with space of linear operators $\mathcal{L}(\mathbb{C}^4)$. For transition B to a new basis in the space $\mathbb{C}^4$, may be written an usual formula of linear algebra for transformation of operators

$$\psi' = B\psi, \quad \psi \in \mathbb{C}^4; \qquad M' = BMB^{-1}, \quad M \in \mathcal{L}(\mathbb{C}^4). \tag{5.4}$$

So, covariant properties of the Dirac operator and four-component spinors are in agreement with the transformations of an operator algebra. It may be also checked directly, the transformation Eq. (5.2) respects the Dirac equation

$$(\mathfrak{S}\mathcal{D}\mathfrak{S}^{-1})\mathfrak{S}\psi = m\mathfrak{S}\psi \tag{5.5}$$

and $\mathcal{D} \mapsto \mathfrak{S}\mathcal{D}\mathfrak{S}^{-1}$ acts on tangent vector (*i.e.* partial derivatives ∂_μ in Dirac operator $\mathcal{D}$) as a Lorentz transformation due to Eq. (5.2).

It should be mentioned, that Eq. (5.4) also describes a formal transition between different matrix representations of the generators γ^k, but it is rather a question of notation and does not have direct relation with the covariance. For example, together with presentation Eq. (1.10) may be also used so-called *standard* one [1, 2]

$$\gamma_s^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_s^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \quad (k = 1, 2, 3) \tag{5.6}$$

Similarly with general case Sec. 2, for any isometry, *i.e.* Lorentz trasformation $L \in O(3, 1)$ there are two ways Eq. (2.15) and Eq. (2.22) to write the same transformation

$$\gamma^k \mapsto \gamma'^k = \sum\nolimits_j L^k_j \gamma^j = \mathfrak{S}_L \gamma^k \mathfrak{S}_L^{-1}. \tag{5.7}$$

It was already discussed in Sec. 2, for general linear transformations of a basis, *i.e.* $G \in GL(4, \mathbb{R})$ it is not possible to write analogue of Eq. (5.7). For description of spinors in General Relativity often is used construction already mentioned in Sec. 2.4. It is introduced a diagonal metric in normalized basis together with a matrix A of transition to initial basis and metric [15, 18]. The 4×4 matrix A may be considered as four vectors called *tetrad* [18, 19, 20, 21].

It was already mentioned, the main theme of present article — is extension of Dirac equation from 4D space of Dirac spinors to 16D linear space, there covariant action of $GL(4, \mathbb{R})$ group may be written directly.

For such a consideration it is convenient to use different algebraic structures on an extended linear space and here is used following sequence of 16D linear spaces

$$\Lambda(\mathbb{C}^4) \cong \mathfrak{Cl}(4, \mathbb{C}) \cong M(4, \mathbb{C}) \cong \mathbb{C}^4 \otimes \mathbb{C}^4. \tag{5.8}$$

It is also possible to consider a real analogue of the sequence Eq. (5.8)

$$\Lambda(\mathbb{R}^4) \cong \mathfrak{Cl}(3, 1) \cong M(4, \mathbb{R}) \cong \mathbb{R}^4 \otimes \mathbb{R}^4. \tag{5.9}$$

5.2 Dirac Equation and Grassmann Algebras

It was already discussed in Sec. 4 that exterior forms have well defined covariant properties with respect to any transformation of coordinate system. For an exterior algebra on the tangent space of 4D manifold $\mathcal{M}$ (the spacetime) we have the decomposition

$$\Lambda(\mathcal{M}) = \Lambda^0(\mathcal{M}) \oplus \Lambda^1(\mathcal{M}) \oplus \Lambda^2(\mathcal{M}) \oplus \Lambda^3(\mathcal{M}) \oplus \Lambda^4(\mathcal{M}) \tag{5.10}$$

with dimensions $\{1, 4, 6, 4, 1\}$ respectively. It corresponds to decomposition of 16D linear space $\Lambda(\mathcal{M})$ on five subspaces of irreducible representations $GL(4,\mathbb{R})$ group with given dimensions. Transformation properties of $\Lambda^k(\mathcal{M})$ corresponds to tensors with type $(0,k)$ and was already considered in Sec. 4, see Eq. (4.2).

It is also convenient to consider transformation properties of $\Lambda(\mathcal{M})$ with respect to $SL(4,\mathbb{R})$ subgroup of the matrices with unit determinant. The group does not change 4-volume form $\Omega = dt \wedge dx \wedge dy \wedge dz$. The Hodge operator $\star$ Eq. (4.9) interchanges $\Lambda^k(\mathcal{M})$ and $\Lambda^{4-k}(\mathcal{M})$ forms. So both $\Lambda^0(\mathcal{M})$ and $\Lambda^4(\mathcal{M})$ corresponds to a trivial, scalar representation of $SL(4,\mathbb{R})$ group, $\Lambda^1(\mathcal{M})$ and $\Lambda^3(\mathcal{M})$ — to 4D vector representation.

Really, because $\star : \Lambda^k(\mathcal{M}) \to \Lambda^{4-k}(\mathcal{M})$ acts as duality, the representation $\Lambda^3(\mathcal{M})$ should be considered as dual to $\Lambda^1(\mathcal{M})$. Due to the same principle, $\star : \Lambda^2(\mathcal{M}) \to \Lambda^2(\mathcal{M})$ is duality of 6D space $\Lambda^2(\mathcal{M})$. A transformation to the dual space is equivalent to the introduction of some metric and the Hodge duality defines a metric on 6D space $\Lambda^2(\mathcal{M})$.

This metric with signature $(3,3)$ is relevant with representation of $SL(4,\mathbb{R})$ group as double cover of $SO(3,3)$ [8], two elements $\pm G \in SL(4,\mathbb{R})$ map to same element $S \in SO(3,3)$, *i.e.* the classical (matrix) group $SL(4,\mathbb{R})$ is isomorphic with $Spin(3,3)$.

It is an analogue of so-called Klein relations for Plücker coordinates and well known from the theory of Lie groups, twistors [10, 14] *etc.*. The general construction for a complex space $\mathbb{C}^4$ and 2→1 map $SL(4,\mathbb{C})$ to $SO(6,\mathbb{C})$ produces by restriction on different real subspaces isomorphisms like $SL(4,\mathbb{R}) \cong Spin(3,3)$, $SU(4) \cong Spin(6)$, $SU(2,2) \cong Spin(4,2)$ *etc.*

Let us consider the space of 2-forms, with coordinates denoted as $p^{jk} dx_j \wedge dx_k$, then the Hodge duality corresponds to invariance of form

$$p^{01}p^{23} + p^{02}p^{31} + p^{03}p^{12}. \tag{5.11}$$

If to introduce 6 new coordinates

$$\begin{aligned} q_1 &= (p^{01} + p^{23})/2, \quad q_2 = (p^{02} + p^{31})/2, \quad q_3 = (p^{03} + p^{12})/2, \\ q_4 &= (p^{01} - p^{23})/2, \quad q_5 = (p^{02} - p^{31})/2, \quad q_6 = (p^{03} - p^{12})/2, \end{aligned} \tag{5.12}$$

the quadratic form Eq. (5.11) may be rewritten as

$$q_1^2 + q_2^2 + q_3^2 - q_4^2 - q_5^2 - q_6^2. \tag{5.13}$$

New basis in the space $\Lambda^2(\mathcal{M})$ corresponding to such a diagonal form may be chosen as

$$\begin{aligned} \eta_1 &= dx_0 \wedge dx_1 + dx_2 \wedge dx_3, \quad \eta_4 = dx_0 \wedge dx_1 - dx_2 \wedge dx_3, \\ \eta_2 &= dx_0 \wedge dx_2 + dx_3 \wedge dx_1, \quad \eta_5 = dx_0 \wedge dx_2 - dx_3 \wedge dx_1, \\ \eta_3 &= dx_0 \wedge dx_3 + dx_1 \wedge dx_2, \quad \eta_6 = dx_0 \wedge dx_3 - dx_1 \wedge dx_2. \end{aligned} \tag{5.14}$$

If a basis of 1-forms $dx_k \in \Lambda^1(\mathcal{M})$, $k = 0, \ldots, 3$ is transformed by element of group $SL(4,\mathbb{R})$, then the basis of 2-forms $\eta_p \in \Lambda^2(\mathcal{M})$, $p = 1, \ldots, 6$ is transformed by some

element of $SO(3,3)$, because it saves Eq. (5.13) invariant, and it is also clear from Eq. (5.14), two elements $\pm G \in SL(4,\mathbb{R})$ correspond to the same transformation.

So action of the group $SL(4,\mathbb{R})$ on 16D space $\Lambda(\mathcal{M})$ is decomposed on trivial representation for 1D spaces $\Lambda^0(\mathcal{M})$ and $\Lambda^4(\mathcal{M})$, 4D "vector" representation $SL(4,\mathbb{R})$ on spaces $\Lambda^1(\mathcal{M})$ and $\Lambda^3(\mathcal{M})$, and 6D representation $SO(3,3)$ on "autodual" space $\Lambda^2(\mathcal{M})$.

Full group $GL(4,\mathbb{R})$ also includes the dilations $\lambda\mathbf{1}$ (where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\mathbf{1}$ is 4×4 unit matrix) and the improper transformations with $\det = -1$, like time inversion. The dilations acts as λ^k on $\Lambda^k(\mathcal{M})$. For any improper transformation there is additional (-1) multiplier for $\Lambda^4(\mathcal{M})$ ("pseudoscalars"), $\Lambda^3(\mathcal{M})$ ("pseudovectors") and quadratic form in Eq. (5.13).

It was already discussed in Sec. 4.2, the Dirac operator on exterior forms $\check{\mathcal{D}} : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{M})$ is defined in covariant way Eq. (4.18)

$$\check{\mathcal{D}} = (d - d^\star), \quad d : \Lambda^k(\mathcal{M}) \to \Lambda^{k+1}(\mathcal{M}), \quad d^\star : \Lambda^k(\mathcal{M}) \to \Lambda^{k-1}(\mathcal{M}). \tag{5.15}$$

Here for definition of the operator $d^\star$ it is necessary to have a metric g on the vector space $V = T\mathcal{M}$ (tangent space of $\mathcal{M}$), and if there are given $\Upsilon \in \Lambda(V)$, g, d and $d^\star$ in one coordinate system, it is possible to write transformation of all four geometrical objects for an arbitrary change of the coordinate system. So if the Dirac-Kähler equation

$$\check{\mathcal{D}}\Upsilon \equiv (d - d^\star)\Upsilon = m\Upsilon, \tag{1.12$'$}$$

was satisfied in one coordinate system, it is also true for any transformation. So Eq. (1.12$'$) is covariant with respect to $GL(4,\mathbb{R})$ group of general linear transformations of coordinate system.

It should be mentioned, both d and $d^\star$ change range of homogeneous k-forms from $\omega \in \Lambda^k(V)$ and so the form $\Upsilon \in \Lambda(V)$ might be *nonhomogeneous* to satisfy Eq. (1.12$'$). Such a form is a sum of few k-forms with different k and it is enough to write transformation for each such a term.

The relation between such a "tensor" form of a Dirac equation and spinoral one is well known [6, 7], but the purpose of given article is to consider *transformation properties* of the equations and a model of Dirac spinor as a *subsystem* in a 16D space. It is convenient first to revisit a model of Dirac spinors as an ideal of Clifford algebra [7, 22].

5.3 Matrix Dirac Equation and Clifford Algebras

Dirac equation on a Clifford algebra may be simply written

$$\mathcal{D}\Psi = m\Psi, \quad \Psi \in \mathfrak{Cl}(4,\mathbb{C}) \cong M(4,\mathbb{C}), \quad \mathcal{D} = \sum_{k=0}^{3} i\gamma^k \frac{\partial}{\partial x_k}. \tag{5.16}$$

Here Ψ is not spinor, but element of the Clifford algebra. The Spin group also may be considered as some subset of the Clifford algebra, so γ^k and Ψ are elements of the same space $\mathfrak{Cl}(4, \mathbb{C})$.

Formally usual Dirac equation Eq. (1.6) may be considered as special form of Eq. (5.16), then the matrix Ψ has only one nonzero column ψ. More algebraic way to describe such a model — is to consider *left ideals* of a Clifford algebra [7].

The left ideal of an algebra $\mathcal{A}$ by definition [23] is a linear subspace $\mathcal{I} \subset \mathcal{A}$ with property $\mathcal{A}\mathcal{I} \subset \mathcal{I}$, *i.e.* any element of the algebra after multiplication on an element of the ideal produces again an element of the ideal. A simplest example of a left ideal in the matrix algebra is a set of matrices with only one nonzero column already mentioned above

$$M_\psi = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \tag{5.17}$$

A specific point of such representation is the transformation properties of given equation. All elements of an algebra should have the same transformation properties, so if to consider Ψ and γ^k as equivalent elements of the same algebra, then even for the Lorentz transformations instead of Eq. (5.5) with $\psi \mapsto \mathfrak{S}\psi$, it is necessary to consider

$$(\mathfrak{S}\mathcal{D}\mathfrak{S}^{-1})\mathfrak{S}\Psi\mathfrak{S}^{-1} = m\mathfrak{S}\Psi\mathfrak{S}^{-1}, \quad \Psi \mapsto \mathfrak{S}\Psi\mathfrak{S}^{-1}. \tag{5.18}$$

Such a difference in a transformation property of Ψ does not produce a serious problem for comparison with the initial equation Eq. (1.6), due to an additional symmetry $\Psi \to \Psi R$ of Eq. (5.16), *i.e.* the right multiplication on the arbitrary element R of the Clifford algebra.

In such a case the right multiplication on $\mathfrak{S}^{-1}$ in Eq. (5.18) is not so essential. For the comparison of Eq. (5.16) with a spinor equation Eq. (1.6), it is necessary to consider not only the element M_ψ Eq. (5.17) of some left ideal, but also all $M_\psi R$, *i.e.* space of matrices with all columns proportional to the same vector.

If $\psi \in \mathbb{C}^4$ is the initial 4-spinor, and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{C}^4$ is an arbitrary vector of coefficients, then instead of Eq. (5.17) in is necessary to consider a matrix with proportional columns, $M_{ij} = \psi_i \alpha_j$,

$$M = \begin{pmatrix} \alpha_1\psi_1 & \alpha_2\psi_1 & \alpha_3\psi_1 & \alpha_4\psi_1 \\ \alpha_1\psi_2 & \alpha_2\psi_2 & \alpha_3\psi_2 & \alpha_4\psi_2 \\ \alpha_1\psi_3 & \alpha_2\psi_3 & \alpha_3\psi_3 & \alpha_4\psi_3 \\ \alpha_1\psi_4 & \alpha_2\psi_4 & \alpha_3\psi_4 & \alpha_4\psi_4 \end{pmatrix}, \quad M = \psi\alpha^T \equiv \psi \otimes \alpha. \tag{5.19}$$

For arbitrary matrices L, R it may be written

$$LMR = L\,(\psi \otimes \alpha)\,R = (L\psi) \otimes (R^T\alpha), \tag{5.20}$$

so the left and right multiplication saves "the product structure" and it will be used further in Sec. 5.4, but space of all matrices in form Eq. (5.19) is not an ideal,

because it is not *linear subspace*, *e.g.* sum of elements does not necessary may be presented again as a tensor product of two vectors Eq. (5.19). Really linear span of the "degenerate" ($\det M = 0$) subspace of all such matrices Eq. (5.19) coincides with the whole matrix algebra.

Yet, fixed $\alpha \in \mathbb{C}^4$ corresponds to *the left ideal*, for example initial construction with one column is the particular case of such an ideal for $\alpha = (1,0,0,0)$. On the other hand, fixed ψ corresponds to a linear subspace, *the right ideal* $\mathcal{R}_\psi$ of the algebra.

Such a procedure also has analogues with consideration of *an equivalence classes* $MR \sim M$, $M \in \mathfrak{Cl}$ [7] and partially with usual spinor case, where different physical states correspond to *the rays in Hilbert space*, *i.e.* equivalence classes $\psi \sim \lambda\psi$. A model for relation between Dirac spinors $\psi \in \mathbb{C}^4$ and the equivalence classes in 16D linear spaces $\mathfrak{Cl}(4,\mathbb{C}) \cong M(4,\mathbb{C}) \cong \Lambda(\mathbb{C}^4)$ is called *Dirac–Hestenes spinors* in [7]. On the other hand, only $SO(3,1)$ group respects the equivalence classes and so such construction is not valid for general coordinate transformations from $GL(4,\mathbb{R})$ [9].

The property should be discussed here with more details. The covariant property of $\Lambda(\mathbb{R}^4)$ or $\Lambda(\mathbb{C}^4)$ discussed in Sec. 5.2 may be extended to other linear spaces $\mathfrak{Cl}(4,\mathbb{C}) \cong M(4,\mathbb{C})$ if to choose an isomorphism of the linear spaces (without algebraic structure) and to use a diagram

$$\begin{array}{ccc} \mathcal{D}\Psi & \longrightarrow & \mathcal{D}'\Psi' \\ \big\downarrow{\scriptstyle \mathfrak{E}} & & \big\uparrow{\scriptstyle \mathfrak{E}^{-1}} \\ \check{\mathcal{D}}\Upsilon & \longrightarrow & \check{\mathcal{D}}'\Upsilon' \end{array} \tag{5.21}$$

Here the map of the linear spaces $\mathfrak{E} : \mathfrak{Cl} \to \Lambda$ for the diagonalized (Lorentz) metric is defined as Eq. (4.17)

$$\mathfrak{E} : \gamma^{i_1} \cdots \gamma^{i_k} \mapsto dx_{i_1} \wedge \cdots \wedge dx_{i_k} \tag{5.22}$$

It is convenient also to choose the particular matrix representation of Clifford algebra say Eq. (1.10), and to consider Eq. (5.16) as a matrix equation, and let $\mathfrak{E}_\mathfrak{m}$ is the map between the linear spaces of 4×4 matrices and exterior forms, corresponding to $\mathfrak{E}$ in the given matrix representation. Then the transformation of the matrix Ψ for arbitrary $G \in GL(4,\mathbb{R})$ may be written

$$G_M : \Psi \mapsto \mathfrak{E}_\mathfrak{m}^{-1}\Big(G_\Lambda\big(\mathfrak{E}_\mathfrak{m}(\Psi)\big)\Big), \tag{5.23}$$

where G_Λ is the action G on the space Λ of the nonhomogeneous forms discussed above in Sec. 5.2.

The Eq. (5.23) sometime is better to consider as a formal equation for a map between two linear spaces without any additional structures, *i.e.* the choice of gamma matrices and bases in $M(4,\mathbb{C})$ and $\Lambda(\mathbb{C}^4)$ represent the map $\mathfrak{E}_\mathfrak{m}$ and G_Λ as some formal linear transformations of $\mathbb{C}^{16}$. The map G_M may be considered as the

composition of the transformations $G_M = \mathfrak{C}_{\mathfrak{m}} G_\Lambda \mathfrak{C}_{\mathfrak{m}}^{-1}$, *viz* product of three 16×16 complex matrices.

Such a "technical" point of view is justified for representation of the general coordinate transformations, because G_M is the isomorphism only for the Lorentz group, but for a general element $G \in GL(4,\mathbb{R})$, it is not necessary in agreement with Clifford multiplication

$$\exists\, G \notin SO(3,1): \quad G_M(\gamma^j)G_M(\gamma^k) \neq G_M(\gamma^j\gamma^k), \tag{5.24}$$

and so attempts to use a definition like Eq. (5.22) for a Clifford algebra with a non-diagonal quadratic form g may produce some problems. Formally Eq. (5.22) should be chosen for a normalized basis.

The inequality Eq. (5.24) appears because only isometries always respect the subspaces of k-multivectors $\mathfrak{Cl}^{(k)}$ Eq. (2.3), but the subspaces of k-forms Λ^k due to Eq. (4.2) are invariant for arbitrary linear transformations. So $G_M(\gamma^{i_1}\cdots\gamma^{i_k}) \in \mathfrak{Cl}^{(k)}$, but $G_M(\gamma^{i_1})\cdots G_M(\gamma^{i_k})$ may include also additional terms from $\mathfrak{Cl}^{(j)}$, $j \leq k-2$.

It is useful to consider some examples.

- Let we have some transformation in $x-y$ plane.

  $$\gamma^1 \mapsto a_{11}\gamma^1 + a_{12}\gamma^2, \quad \gamma^2 \mapsto a_{21}\gamma^1 + a_{22}\gamma^2, \tag{5.25}$$

  then the multivector $\gamma^1\gamma^2$ is transformed as

  $$\gamma^1\gamma^2 \mapsto -(a_{11}a_{21} + a_{12}a_{22})1 + (a_{11}a_{22} - a_{12}a_{21})\gamma^1\gamma^2$$

  So a rotation $a_{11} = a_{22} = \cos\varphi$, $a_{12} = -a_{21} = \sin\varphi$ maps $\gamma^1\gamma^2$ to itself, but for a nonorthogonal transformation an additional term with unit appears. On the other hand, for the 2-form $dx_1 \wedge dx_2$ such a term may not appear, because $dx_1 \wedge dx_1 = dx_2 \wedge dx_2 = 0$. Only for rotations $\gamma^1\gamma^2$ and $dx_1 \wedge dx_2 = \mathfrak{E}(\gamma^1\gamma^2)$ have the same transformation properties.

- On the other hand, for a transformation in $t-x$ plane

  $$\gamma^0 \mapsto b_{00}\gamma^0 + b_{01}\gamma^1, \quad \gamma^1 \mapsto b_{10}\gamma^0 + b_{11}\gamma^1, \tag{5.26}$$

  then the multivector $\gamma^0\gamma^1$ is transformed as

  $$\gamma^0\gamma^1 \mapsto (b_{00}b_{10} - b_{01}b_{11})1 + (b_{00}b_{11} - b_{10}b_{01})\gamma^0\gamma^1$$

  So, the boosts $b_{00} = b_{11} = \cosh\upsilon$, $b_{10} = b_{01} = \sinh\upsilon$ map $\gamma^0\gamma^1$ to itself, *i.e.* again only the isometries of the concrete quadratic form (the pseudo-Euclidean Minkowski metric) respect the subspaces of the Clifford k-multivectors ($\mathfrak{Cl}^{(2)}$ in given examples).

5.4 Tensor Product of Two Spinor Spaces

For any element of Lorentz group $L \in O(3,1)$ exists spinor representation Eq. (5.18) of transformation Eq. (5.23), *i.e.*

$$\forall L \in O(3,1),\ \exists \mathfrak{S}_L : \quad L_M(\Psi) = \mathfrak{S}_L \Psi \mathfrak{S}_L^{-1}. \tag{5.27}$$

On the other hand, any linear transformation of the 16D space of 4×4 matrices may be represented as

$$\Psi \to \sum_J A_J \Psi B_J, \tag{5.28}$$

there A_J, B_J are some matrices. So for arbitrary $G \in GL(4,\mathbb{R})$ the transformation G_M also may be represented using Eq. (5.28), but only for Lorentz transformation such representation has only one term Eq. (5.27).

It has an analogue with idea of representation $\mathfrak{Cl}$ as tensor product of two spinor spaces $S \otimes S$ in general algebraic theory [17]. For present discussion it is enough to consider concrete model with algebra of 4×4 complex matrices. It corresponds to last term

$$M(4,\mathbb{C}) \cong \mathbb{C}^4 \otimes \mathbb{C}^4 \tag{5.29}$$

in sequence of 16D linear spaces Eq. (5.8) already mentioned briefly Sec. 5.3 in relation with treatment of solution of the matrix Dirac equation Eq. (5.16) as a tensor product of two complex 4-vectors Eq. (5.19).

Due to Eq. (5.27), for a transformation L from Lorentz group Eq. (5.20) may be rewritten as

$$L_M(\psi \otimes \alpha) = (\mathfrak{S}_L \psi) \otimes ({\mathfrak{S}_L^{-1}}^T \alpha), \quad \psi, \alpha \in \mathbb{C}^4, \tag{5.30}$$

but for a general linear transformation $G \in GL(4,\mathbb{R})$ it is necessary to use Eq. (5.28) with some set $\mathfrak{L}_J(G)$, $\mathfrak{R}_J(G)$

$$G_M(\psi \otimes \alpha) = \sum_J (\mathfrak{L}_J \psi) \otimes (\mathfrak{R}_J \alpha), \quad \psi, \alpha \in \mathbb{C}^4, \tag{5.31}$$

Technically the Eq. (5.31) even not guarantee the existence of an unique decomposition and may look more complicated, than the initial equation Eq. (5.23). Method of calculation Eq. (5.23) is difficult, but straightforward, it would be possible to write precise expressions for any choice of isomorphism of 16D linear spaces $\Lambda(\mathbb{C}^4)$ and $M(4,\mathbb{C}) \cong \mathbb{C}^4 \otimes \mathbb{C}^4$ (*c.f.* [6, 7]) using tedious calculations or cumbersome output of some computer algebra system, but it most likely would not clarify a physical properties of the transformation. On the other hand, the Eq. (5.31) may be useful for some intuitive explanation [9] used in further discussion.

5.5 A Formal Model with Two Quantum Systems

In the quantum mechanics the state space of a compound quantum system may be described as the tensor product of the spaces of each subsystem. So, formally due to Eq. (5.29) Ψ may be considered as a composite system with two particles. The parts are called *not entangled* (uncorrelated), if the state such a system may be represented as a tensor product $\psi_1 \otimes \psi_2$ [24]. So, in Eq. (5.19) Ψ is a formal analogue of such non-entangled system $\psi \otimes \alpha$. It is clear also, that due to Eq. (5.30) a Lorentz transformations respect the entanglement. *The correspondence between Ψ and ψ is an analogue of relation between system and subsystem in the quantum mechanics.*

It should be mentioned, that in the constructions used below $\psi(x)$ is a function of the point x, but α is a constant complex 4-vector. Formally Ψ corresponds to the two uncorrelated systems, ψ and α. The first system satisfies to Dirac equation. The second one formally also satisfies Dirac equation for a massless particle, but a state of the system does not matter: it is not correlated with the first one, can be considered as an auxiliary system and a Lorentz transformation may not change such a structure. So, if to consider only the Lorentz transformations, the matrix Dirac equation Eq. (5.16) is equivalent with usual one Eq. (1.6).

On the other hand, a general coordinate transformation due to Eq. (5.31) does not preserve the property of being non-entangled. Formally we may reduce the matrix Dirac equation Eq. (5.16) to usual one Eq. (1.6) by restriction to an equation for the arbitrary row of Ψ, but after the general coordinate transformation each column of Ψ contains combinations of ψ_i with different α_j, and so depends on the state of the second system. It differs from the case with Lorentz transformations, then any combination of α_j for any particular column due to Eq. (5.30) appears as a common scalar multiplier and always may be omitted.

In the quantum mechanics there is a standard method of consideration of a subsystem of an entangled system — the mixed states and the density matrix [24, 25, 26]. It should be recalled, that all the structures discussed here in relation with the formal decomposition $\Psi \in S \otimes S$ yet have *rather mathematical* resemblance with the quantum mechanics and was used mainly with an illustrative reason, but the appearance of some analogue of the mixed states directly from the properties of the group $GL(4, \mathbb{R})$ of the general linear transformation of coordinate systems, may be promising.

6 Curved Space-Time and Singularities

6.1 Dirac Operator and General Relativity

Despite of using an arbitrary metric and the general linear transformation of a coordinate system, the vast amount of the material presented above was relevant rather with the flat space-time. The applications of the Dirac operator in a curved

space may depend on some implicit suggestions. For example for the Euclidean metric it is possible to calculate different kinds of local Laplacians $\mathcal{D}^2$: spinor, Hodge or twisted one [3]. All such Laplacians contain a term with scalar curvature $\frac{1}{4}R(x)$. The Hodge and twisted spinor Laplacians also contain additional terms with sums of fourth order [3]. Such a kind of terms could produce a specific kind of the Klein–Gordon operator in the curved space time like $\Box \pm \frac{1}{4}R + m^2$, and it does not look as a proper physical one.

It was already mentioned that most calculations with the spinors in General Relativity use the various applications of tetrad formalism or some analogues. For example the Dirac equation in the Newman–Penrose formalism for a charged black hole described by the Reissner–Nordstörm metric and for a rotated one with the Kerr metric may be found in [20].

On the other hand, there is some formal question: *it is known, that the black hole evaporation may be related with a transition from a pure state to a mixed state of a quantum system* [18, 27, 28, 29]*, but how in principle to consider such a transition?.*

It was mentioned at end of Sec. 5.4, that analogue of such transition has some allusions with consideration of the general linear coordinate transformation on the extended (matrix) Dirac equation Eq. (5.16).

Of course, there are huge amount of works and different approaches to black hole evaporation. It is necessary to use methods of field theory for rigour research [18, 30, 31, 32], but such methods in very core always use covariance with respect to group of coordinate transformations, an so it is anyway necessary to consider structure of such a group first, for using theory of infinite-dimensional unitary representations [34, 33]. Theory of infinite-dimensional representation of $GL(4,\mathbb{R})$ and affine groups in relation with generalization of Dirac equation is also discussed in [35].

It should be mentioned also, that here is considered property of single system. In some works like [30] transition from pure to mixed state in black hole, "information loss paradox" is considered as an analogue of process with "intervention of classical world" in modern applied theory of open quantum systems[3], but in such a case the process of evaporation of black holes lost the uniqueness attracting much attention. Introduction of density matrix, *i.e.* statistical operator in very beginning suggests incomplete description of quantum systems [19, 26] and so in such a case instead of an answer the question partially transferred in set of postulates.

On the other hand, the considered model of the matrix Dirac equation justifies the phenomenon of transition from pure to mixed state not for wide variety of conditions, but only in presence of a singularity (see Sec. 6.2 below), and so may provide the subtler classification. Despite of the technical differences, here the treation of mixed state resembles [36], *i.e.* it *does not suggest* with very beginning neither the work with statistical ensembles, nor interaction with environment, nor any ideas about relation between quantum and classical world, *etc.* The lost of information in

[3]Convenient for practical applications, but partially empirical, with lack of agreement on some fundamental theoretical questions.

present model is due to consideration of only four components ψ in 4×4 matrix Ψ.

6.2 Connections, Gauge Theories and Gravitation

The problem of the consideration of spaces with singularities is well known [37]. If such a problem may have some relevance with a transition from a pure to a mixed state due to usage of an extended covariant equation Eq. (5.16) and $GL(4,\mathbb{R})$ group discussed above?

For a space with a singularity there is a problem with definition of metric on some subspaces. The (pseudo-)Riemannian manifold is the particular example of the affinely connected space [16]. The affine or linear connection and the metric — are two different geometrical objects, it is possible formally to consider a manifold with linear connection without any metric at all[4].

The affinely connected space also is the particular example of a more general theory of connections on a principle bundle with an arbitrary structure group, but this theory has some counterpart in the physics — *the gauge theory.* For example such a link is especially transparent in *Ashtekhar's (loop) quantum gravity* [21], *Ivanenko's gauge theory of gravity* [39], but many methods may be useful for description of different structures in arbitrary models of gravitation, including a "canonical" one, higher-dimensional and super-gravitation [40].

In the general theory, it is considered a connection on *a principal bundle* with some Lie group [16]. For a linear connection the structure group is $GL(4,\mathbb{R})$, for affine one — the affine group $A(4)$, but it is possible to work with the linear connection on a manifold instead of affine one without lost of generality [16]. The important question — is the reduction of the structure group to some subgroup.

For General Relativity, it is the reduction to the Lorentz group $SO(3,1)$ — it is just the question, if it is possible to choose *the atlas*, there all transition between different maps may be described by transformations from the considered subgroup [16]. In General Relativity it is related with the question about possibility to use only the transformation from the Lorentz group for transition between different coordinate systems and due to a general theorem of differential geometry [16], it is possible iff *globally* exist the Minkowski metric and the tetrad field — it is one geometrical treatment of *the equivalence principle in General Relativity* [39, 40].

The consideration above shows, *that the questions about a singularity of the metric and about impossibility of the reduction from the general coordinate transformation to Lorentz one are directly related.* Of course there is a problem, if the physical singularity does not accept also introduction of an arbitrary group of transformation, even more general than Lorentz group. A "toy model" discussed below in Sec. 6.3 may be useful from such a point of view.

[4] It is should not be treated as a "flat" space, because the conception of flatness or curvature is not possible to introduce without a metric.

6.3 Schwarzschild Spacetime

Initial metric of Schwarzschild spacetime may be represented in polar coordinates as [19, 20, 37]

$$ds^2 = (1 - r_g/r)dt^2 - (1 - r_g/r)^{-1}dr^2 - r^2 dS^2, \quad r > r_g \tag{6.1}$$

where $dS^2 = d\theta^2 + d\varphi^2 \sin^2(\theta)$ is the usual spherical surface metric and r_g is a constant (the gravitational radius). If the metric used for the description of the gravitation field outside of a spherical body, it should be chosen $r_g = 2\kappa m/c^2$ to satisfy a Newtonian limit [19]. Let us "rescale" $r_g = 1$ for convenience.

$$ds^2 = \frac{r-1}{r}dt^2 - \frac{r}{r-1}dr^2 - r^2 dS^2, \quad r > 1. \tag{6.1$'$}$$

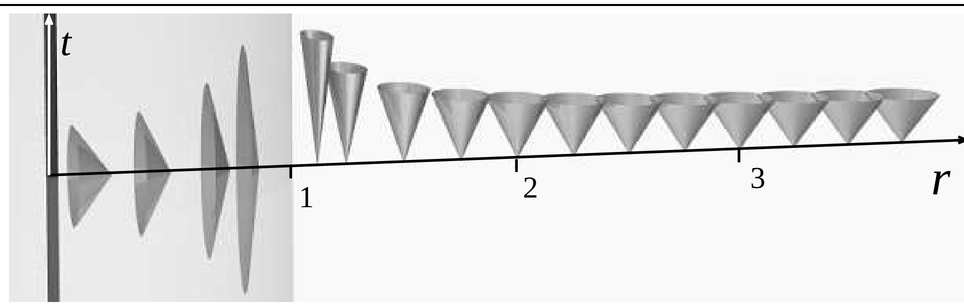

Figure 2: Light cones for Schwarzschild metric.

If try to extend the metric Eq. (6.1) for arbitrary values $r \geq 0$, it has singularities at $r = 0$ and $r = 1$. The usual practice is to consider a question about the isometric embedding of the space $r > 1$ as a some subspace of a bigger manifold [37].

Here is useful to recall a wide family of such embeddings described as [19]

$$\tau = \pm t \pm \int \frac{r f(r)}{r-1} dr, \quad R = t + \int \frac{r}{(r-1)f(r)} dr, \tag{6.2}$$

there $f(r)$ is some function of r. For example the Eddington–Finkelstein coordinates is described [37, 38]

$$v = t + \tilde{r}, \quad w = r - \tilde{r}, \quad \tilde{r} \equiv \int \frac{r}{r-1} dr = r + \ln(r-1), \tag{6.3}$$

formally corresponding to the choice $f(r) \equiv 1$. For such a case in the coordinates (v, r, θ, φ) the metric may be written

$$ds^2 = \frac{r-1}{r}dv^2 - 2dr\,dv - r^2 dS^2. \tag{6.4}$$

In [19] is chosen $f(r) = 1/\sqrt{r}$, because in such a case the coordinate system $(\tau, R, \theta, \varphi)$ is synchronous $(g_{\tau\tau} = 1)$.

It is convenient here to use yet another choice: $f(r) = \sqrt{2r-1}/r$, then Eq. (6.2) produce

$$\begin{aligned} \tau &= t - \int \frac{\sqrt{2r-1}}{r-1}\, dr = t - 2\sqrt{2r-1} + \ln\Bigl(\frac{\sqrt{2r-1}+1}{\sqrt{2r-1}-1}\Bigr), \\ R &= t + \int \frac{r^2\, dr}{(r-1)\sqrt{2r-1}} = t + \frac{r+4}{3}\sqrt{2r-1} - \ln\Bigl(\frac{\sqrt{2r-1}+1}{\sqrt{2r-1}-1}\Bigr). \end{aligned} \tag{6.5}$$

It is possible similar with Eq. (6.4) to write a metric Eq. (6.1′) in the "mixed" ("skew") coordinates $(\tau, r, \theta, \varphi)$, $dt = d\tau + \sqrt{2r-1}/(r-1)\, dr$

$$ds^2 = \frac{r-1}{r}(d\tau^2 - dr^2) + \frac{2\sqrt{2r-1}}{r} dr d\tau - r^2 dS^2, \quad r \geq \frac{1}{2}. \tag{6.6}$$

The metric Eq. (6.6) may be rewritten as

$$(\cos\vartheta(r) d\tau + \sin\vartheta(r) dr)^2 - (\cos\vartheta(r) dr - \sin\vartheta(r) d\tau)^2 - r^2 dS^2, \tag{6.7}$$

where

$$\vartheta(r) = \operatorname{arccosec}(\sqrt{2r}), \qquad \sin\vartheta(r) = \frac{1}{\sqrt{2r}}, \quad \cos\vartheta(r) = \frac{\sqrt{2r-1}}{\sqrt{2r}}. \tag{6.8}$$

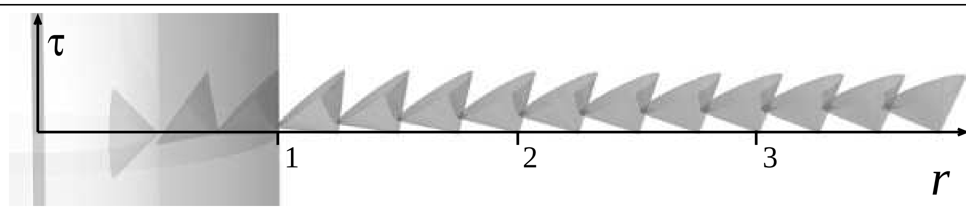

Figure 3: Light cones in Eq. (6.7) metric.

The transformation of a local frame

$$d\tau \mapsto (\cos\vartheta d\tau + \sin\vartheta dr), \quad dr \mapsto (\cos\vartheta dr - \sin\vartheta d\tau), \tag{6.9}$$

used in Eq. (6.7) just corresponds to utilization of the linear transformations more general, than Lorentz one. In Eq. (5.26) Sec. 5.3 below already was considered an example of a general linear transformations in $t-x$ plane. The Eq. (6.7) is similar with the case, then instead of the isometric transformations in tangent space, *i.e.* boost from $SO(3,1) \subset GL(4,\mathbb{R})$ with $b_{00} = b_{11} = \cosh\upsilon$, $b_{10} = b_{01} = \sinh\upsilon$, it is considered an element of $SO(4) \subset GL(4,\mathbb{R})$ with $b_{00} = b_{11} = \cos\vartheta$, $b_{10} = -b_{01} =$

$\sin\vartheta$. It is always exist reduction of $GL(4,\mathbb{R})$ to maximal compact subgroup $SO(4)$ [16, 40].

It is important to mention, that despite of possibility to introduce a local Lorentz metric for any point, it does not possible to do it globally, because it must change sign at some moment (R "becomes time" instead of τ) and it is responsible for appearance of horizon and discontinuity for metric Eq. (6.6) in diagonal form[5] with $(\tau, R, \theta, \varphi)$

$$ds^2 = \frac{r(\tau,R)}{r(\tau,R)-1}\left(d\tau^2 - \frac{2r(\tau,R)-1}{r^2(\tau,R)}dR^2\right) - r^2(\tau,R)\,dS^2, \quad r(\tau,R) \geq \frac{1}{2}. \quad (6.10)$$

The coordinates Eq. (6.5) was chosen to write a simpler example of general linear coordinate transformation. It could be possible to choose some other function $f(r)$, to prevent limitation $r \geq 0.5$, but really presented metric may be defined globally. It is enough to merge two asimptotically flat spaces

$$ds^2 = \frac{r-1}{r}(d\tau^2 - dr^2) \pm \frac{2\sqrt{2r-1}}{r}drd\tau - r^2dS^2, \quad r \geq \frac{1}{2} \quad (6.11)$$

and produce an analogue of *wormhole* [41, 42, 43]. Formally in such a case we let wider diapason of "Euclidean rotation" Eq. (6.9) $0 \leq \vartheta \leq \pi$ instead of $0 \leq \vartheta \leq \pi/2$ for in initial metric Eq. (6.6).

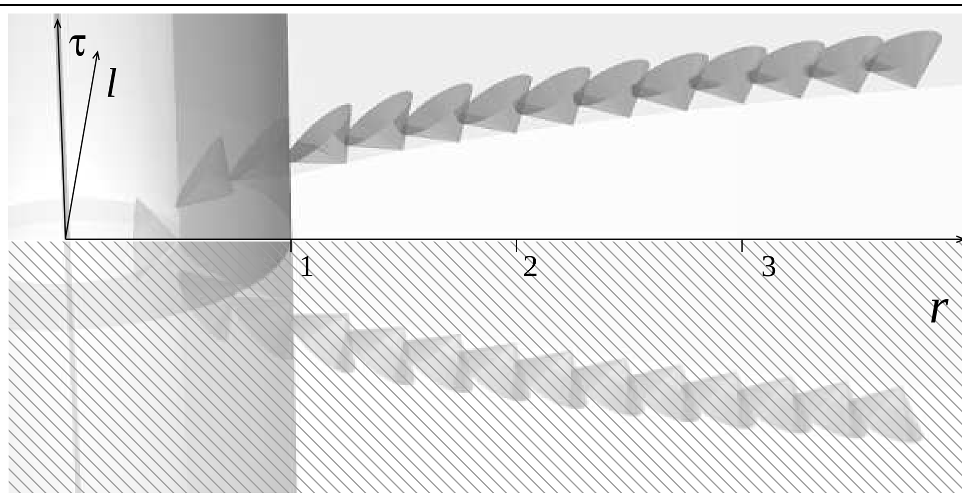

Figure 4: Light cones for wormhole metric Eq. (6.11), Eq. (6.12). A try to show five-dimensional embedding $(\tau, r, l, \theta, \varphi)$ on a two-dimensional picture (see also color picture on the title page).

[5] *C.f.* expression for general $f(r)$ in [19].

The model of wormhole Eq. (6.11) does not look smooth for $r = 1/2$, but it is possible to consider a formal embedding of the four-dimensional manifold $(\tau, r, \theta, \varphi)$ in a five dimensional one $(\tau, r, l, \theta, \varphi)$, where $r = (l^2+1)/2$, $l = \pm\sqrt{2r-1}$ and show, that it is simply due to a problem with a choice of the coordinate r "perpendicular to the throat of the wormhole." Using coordinates $(\tau, l, \theta, \varphi)$, it is possible to rewrite Eq. (6.11)

$$ds^2 = \frac{l^2-1}{l^2+1}(d\tau^2 - l^2 dl^2) \pm \frac{4l^2}{l^2+1} dl d\tau - \frac{(l^2+1)^2}{4} dS^2 \tag{6.12}$$

Really such a kind of metric usually is not considered as a "true" wormhole, because it connects two different "universes," but not domains of the same space. It should be mentioned also, that direction of "proper time" axis τ in second asymptotically flat region is opposite to the first one, it is denoted by "$\pm$" sign in Eq. (6.11). It either produces some difficulty for identification of two flat regions, or illustrate an idea about a close bond between wormholes and time machines [41, 42]. Such kind of *acausal* processes also may be related with unitarity violation [36, 44], but discussion about the theory of wormholes, time machines, *etc.* is not a purpose of presented article.

7 Conclusion

In present article is considered a possibility to write the Dirac equation covariant with respect to the group of general linear coordinate transformations. It was not considered neither infinite-dimensional unitary representations nor methods of the quantum field theory, but even model of matrix Dirac equation with finite-dimensional (16D) space was produced an interesting result. It was shown, that the matrix Dirac equation formally may be considered as an equation for two particles and the general linear coordinate transformation produce specific "entanglement" unlike of Lorentz ones.

The initial Dirac spinor may be considered as a formal subsystem and after general linear coordinate transformation the state becomes entangled with second system, *i.e.* for description it is necessary to use some analogue of *mixed state.* To fasten on a possible relation with the blackhole entropy, in last section was considered the Scwhartchild metric and it was shown, that using of the general linear coordinate transformations may be appropriate for such a case.

It should be mentioned also, that currently the set of questions about relation of black hole thermodynamics, information theory and quantum mechanics in state of active development, *e.g.* already at the time of finishing of given article (March 2004) appear few fresh works with different approaches [31, 32, 45, 46].

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